Liquidity, equilibrium and asymmetric information

Umut Çetin Albina Danilova

London School of Economics, UK

1. To formulate the model of the market precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness. Assume that on this probability space there exist a continuous random variable $V \in \mathcal{F}_0$ and a standard Brownian motion B, independent of V.

We consider a market in which a single risky asset is traded. The value of this asset, V, will be a public knowledge at some future time $t = 1$. For simplicity of exposition we assume that the risk free interest rate is 0.

There are three types of agents that interact in this market:

i) Liquidity traders, whose demands are random, price inelastic and do not reveal any information about the value of V. In particular we assume that their cumulative demand at time t is given by $Z_t = \sigma B_t$.

ii) A single insider who knows V at time $t = 0$ and is risk neutral. We will denote insider's cumulative demand at time t by X_t. The filtration of the insider, $\mathcal{F}^I$, is generated by observing the price of the risky assset and V.

iii) Market makers observe the net supply of the risky asset, $Y = X + Z$, thus, their filtration, $\mathcal{F}^M$, is generated by Y.
We also assume that the market makers have identical CARA utilities with the common risk aversion parameter ρ, and compete in a Bertrand fashion for the net supply of the risky asset. In case of several market makers quoting the same winning price, we adopt the convention that the total order is equally split among them. As a result of this competition in the equilibrium each market maker quotes the price which achieves zero utility gain and, therefore, Y is split equally. The number of market makers is assumed to be $N \geq 2$.

2. The assumption that the markets makers observe only the net supply implies that they cannot separate the informed and uninformed trades. Hence, their quotes at time t can only depend on $(Y_s)_{s=0}^t$. However, we would be looking at only Markovian equilibrium, thus, we consider only the quotes of the form $H(t, Y_t)$. Additionally, we assume that H is smooth enough, i.e. $H(t, y) \in C^{1,2}$, it is strictly increasing in y, and satisfies

$$\mathbb{E}H^2(1, Z_1) < \infty \qquad \text{and} \qquad \mathbb{E}\int_0^1 H^2(t, Z_t)\, dt < \infty. \tag{1}$$

The class of such functions is denoted with $\mathcal{H}$ and any $H \in \mathcal{H}$ is called a *pricing rule.*

As any $H \in \mathcal{H}$ is invertible, observing price is equivalent to observing Y and therefore the insider can perfectly infer the demand of the liquidity traders since she

Speaker: Umut Çetin, e-mail: u.cetin@lse.ac.uk

knows her own demand. It follows from this consideration that $\mathcal{F}_t^I = \sigma(V, Z_s; s \leq t)$. Obviously, for an insider strategy, X, to be admissible, it has to be adapted to $\mathcal{F}^I$. We further require that for a given $H \in \mathcal{H}$

$$\mathbb{E}^v \int_0^1 H^2(t, X_t + Z_t)\, dt < \infty, \tag{2}$$

where $\mathbb{E}^v$ is the expectation taken with respect to $\mathbb{P}^v$ associated to the insider given the realisation $V = v$. Moreover, X is absolutely continuous , i.e. $X_t = \int_0^t \alpha_s\, ds$ (This restriction to the set of absolutely continuous strategies is without loss of generality since strategies with a martingale component and/or jumps are strictly suboptimal as shown in [1]). For any given $H \in \mathcal{H}$ a strategy satisfying the above conditions is called admissible and the class of admissible strategies will be denoted by $\mathcal{A}(H)$. Observe that if $X \in \mathcal{A}(H)$ then the terminal wealth of the insider is given by

$$W_1^X := \int_0^1 X_s dH(s, Y_s) + X_1(V - H(1, X_1)) = \int_0^1 (V - H(s, X_s))\, dX_s. \tag{3}$$

3. By an equilibrium we mean a pair (H^*, X^*) for $H^* \in \mathcal{H}$ and $X^* \in \mathcal{A}(H^*)$ such that

i) given H^*, the insider's strategy X^* solves her optimization problem:

$$\mathbb{E}^v[W_1^{X^*}] = \sup_{X \in \mathcal{A}(H^*)} \mathbb{E}^v[W_1^X].$$

ii) Given X^*, the pricing rule H^* satisfies zero-utility gain condition, i.e. $(U\,(G_t))_{t=0}^1$ is a $(\mathcal{F}^M, \mathbb{P})$-martingale, where

$$G_t := -\frac{1}{N} \int_0^t Y_s^*\, dH^*(s, Y_s^*) + \mathbf{1}_{t=1} \frac{Y_1^*}{N} (H^*(Y_1^*, 1) - V).$$

It is shown that in equilibrium the insider drives the total demand so that $H(1, Y_1) = V$, i.e. the market price converges to the true price. Given this observation, the existence of equilibrium will follow from the following theorem:

Theorem 1. *Suppose that $V = f(\eta)$, where η is a standard normal random variable and f is a strictly increasing function which is either linear or bounded with a continuous derivative. Then, there is a pair (H, Y) which solves the following system:*

$$H_t + \frac{1}{2}\sigma^2 H_{yy} = 0 \tag{4}$$

$$d\xi_t = \sigma dB_t - \frac{\sigma^2 \rho}{2N} \xi_t H_y(t, \xi_t) dt \tag{5}$$

$$V \stackrel{d}{=} H(1, \xi_1), \tag{6}$$

where $\overset{d}{=}$ *stands for equality in distribution.*

Moreover, H_y *is bounded with* $0 < H_y(t, y)$ *for all* $(t, y) \in [0, 1] \times \mathbb{R}$ *and* ξ *is the unique strong solution of (5). Furthermore,* ξ *admits a transition density* $p(s, y; t, z)$ *such that, for any fixed* (t, z)*,* $p(s, y; t, z) > 0$ *on* $[0, t) \times \mathbb{R}$ *and is* $C^{1,2}([0, t) \times \mathbb{R})$*.*

The optimal demand of the insider and the pricing rule in equilibrium are given in the following theorem.

Theorem 2. *Suppose that* $V = f(\eta)$*, where* η *is a standard normal random variable and* f *is either linear or a bounded function with a continuous derivative. Then,* (H^*, X^*) *is an equilibrium where*

$$X_t^* = \int_0^t \left\{ -\frac{\sigma^2 \rho}{2N} Y_s^* H_y^*(s, Y_s^*) + \sigma^2 \frac{p_y}{p}(s, Y_s^*; 1, H^{*-1}(1, V)) \right\} ds$$

and H^* *and* p *are the functions defined in Theorem 1.*

Moreover, under $\mathcal{F}^M$ *the equilibrium demand evolves as*

$$Y_t^* = \sigma B_t^Y - \frac{\sigma^2 \rho}{2N} \int_0^t Y_s^* H_y^*(s, Y_s^*)\, ds.$$

In particular, when $f(y) = ay + b$*, then* $H^*(t, y) = \lambda y + b$*, where* λ *is the unique solution to*

$$1 - e^{-\frac{\rho\sigma^2}{N}\lambda} = \frac{\rho a^2}{N} \frac{1}{\lambda}, \tag{7}$$

and the equilibrium demand Y^* *solves*

$$dY_t^* = \sigma dB_t + \frac{\rho\sigma^2}{2N} \frac{a\eta - \lambda Y_t^* \cosh\left(\frac{\rho\sigma^2\lambda}{2N}(1-t)\right)}{\sinh\left(\frac{\rho\sigma^2\lambda}{2N}(1-t)\right)}. \tag{8}$$

4. In this talk I will also discuss the effect of risk aversion of market makers on various liquidity parameters such as depth and resilience. The comparison with the corresponding features of the model considered in [1] will also be given.

References

[1] Back, K. (1992): Insider trading in continuous time. *The Review of Financial Studies*, 5(3), pp. 387–409.