Expected utility maximization in exponential Lévy models

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1. In modern financial mathematics the problem of maximizing expected utility of the asset portfolio has become increasingly popular.

In our work we consider a model of a financial market with one asset and a finite maturity T. The capital process has the form $X = x + H \cdot S$, where x is the initial wealth, H is a predictable process (also called "strategy") and S is a semimartingale that models the asset's price. We assume that all capital processes belong to the set $\mathcal{X}(x) = \{X_t \geq 0 : X_0 = x\}$. Our aim is to maximize the expected logarithmic utility at time T:

$$u(x) = \sup_{X \in \mathcal{X}(x)} E[\ln(X_T)].$$

Here we suppose that $u(x) < +\infty$.

The problem of utility maximization was considered in [1] by Kramkov and Schachermayer in a general model of incomplete markets and a general utility function finite on $\mathbb{R}_+$. The solution was found by solving the dual problem, where the minimum was taken over the set of supermartingale deflators, not only equivalent local martingale measures.

In our work we consider the case of exponential Lévy models when S is the stochastic exponential of a Lévy process L, $\Delta L \geq -1$, with the triplet (b, c, ν). The problem of maximizing logarithmic utility in exponential Lévy models was solved by Hurd [2] under the assumption that the logarithms of price processes have jumps unbounded from above and below. He used the dual method and indicated that there are cases where the solution of the dual problem is a supermartingale and not necessary a martingale.

It is well known that the solution X^* is the numéraire portfolio and the solution of the dual problem satisfies $X^* Y^* = 1$. Kardaras [3] showed that the numéraire portfolio exists in an exponential Lévy model iff the process L is not monotonous. Recall [4] that the Lévy process is monotonous iff $c = 0,\ \nu[x < 0] = 0,\ b - \int x \mathbb{1}_{|x| \leq 1} \nu(dx) \geq 0$ or $c = 0,\ \nu[x > 0] = 0,\ b - \int x \mathbb{1}_{|x| \leq 1} \nu(dx) \leq 0$. We show that if the monotonous assumption is not satisfied, the numéraire portfolio X^* exists and there are only three possibilities for $Y^* = 1/X^*$.

1. Y^* is a supermartingale, but not a martingale.
2. Y^* is a martingale, but not the density process of an equivalent σ-martingale measure.
3. Y^* is the density process of an equivalent martingale measure.

The aim of our work is to classify all these cases in terms of the Lévy triplet.

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2. Consider the set $\mathfrak{C} = \{p : \nu\{x : (1 + px) < 0\} = 0\}$ and put $\overline{M} = \inf(\mathfrak{C})$, $\overline{N} = \sup(\mathfrak{C})$. Denote by L_1 and L_2 the following quantities:

$$L_1 = c\overline{N} - b + \int_{|x|\le 1} \frac{x^2}{|(1/\overline{N}) + x|}\nu(dx) - \int_{x>1} \frac{x}{1+\overline{N}x}\nu(dx)$$

$$L_2 = c\overline{M} - b + \int_{|x|\le 1} \frac{-x^2}{|(1/\overline{M}) + x|}\nu(dx) - \int_{x>1} \frac{x}{1+\overline{M}x}\nu(dx)$$

Here we use the rules: $0 \cdot \infty = 0,\ 1/\infty = 0,\ 1/0 = \infty$.

Theorem 1. *In a finite-time exponential Lévy model, for $Y^* = 1/X^*$, where X^* is the numéraire portfolio, the following holds true:*

1. *Y^* is the density process of an equivalent martingale measure in one of the following 3 cases:*

   (i) $$b + \int_{x>1} x\nu(dx) > 0 \ \textit{or} + \infty \ \textit{and}$$ $$L_1 \ge 0 \ \textit{if} \ \overline{N} < +\infty \ \textit{or} \ L_1 > 0 \ \textit{if} \ \overline{N} = +\infty.$$

   (ii) $$b + \int_{x>1} x\nu(dx) = 0.$$

   (iii) $$b + \int_{x>1} x\nu(dx) < 0 \ \textit{and}$$ $$L_2 \le 0 \ \textit{if} \ \overline{M} > -\infty \ \textit{or} \ L_2 < 0 \ \textit{if} \ \overline{M} = -\infty.$$

2. *Y^* is a martingale but not the density process of an equivalent martingale measure, when the following is satisfied:*
   $$b + \int_{x>1} x\nu(dx) < 0 \quad \textit{and} \ \overline{M} = 0.$$

3. *Y^* is a supermartingale and not a martingale in one of the following 2 cases:*

   (i) $$b + \int_{x>1} x\nu(dx) > 0 \ \textit{or} + \infty,$$ $$L_1 < 0 \ \textit{and} \ \overline{N} < +\infty.$$

   (ii) $$b + \int_{x>1} x\nu(dx) < 0,$$ $$L_2 > 0 \ \textit{and} \ -\infty < \overline{M} < 0.$$

References

[1] D. Kramkov, W. Schachermayer (1999). The condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. Annals of Applied Probability, Vol. 9, No. 3:904–950.

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[4] K.-I. Sato (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, UK, Cambridge.