From sequential analysis to optimal stopping – revisited

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1. The talk will cover three topics:

1) Sequential testing and change point detection
2) Optimal stopping of diffusions using harmonic functions
3) Determination of saddle points of stopping games

We give two examples which describe our viewpoint. Starting from the classical Wald sequential probability ratio test we elaborate a structure, which is present in many Bayes testing problems.

2. At first we consider the problem of testing the sign of the drift of Brownian motion W for the simple hypothesis H_0: $-\theta$ versus H_1: $+\theta$ with $\theta > 0$. We assume $0-1$ loss and observation cost c per unit time. The Bayes risk for the prior $\frac{1}{2}\delta_{-\theta} + \frac{1}{2}\delta_\theta$ is then defined as

$$R(T,\delta) := \frac{1}{2}\left(P_{-\theta}[H_0 \text{ rejected } (\delta)] + cE_{-\theta}T\right) + \frac{1}{2}\left(P_\theta[H_1 \text{ rejected } (\delta)] + cE_\theta T\right).$$

The goal is to find (T^*, δ^*) which minimize this risk. Let δ^*_T denote the decision rule, which rejects H_0 when $W_T > 0$. Then $R(T, \delta^*_T) \leq R(T, \delta)$ holds for all decision rules δ and stopping times T. Then one can show

$$R(T, \delta^*_T) = \int h(\theta|W_T|)dQ, \qquad (*)$$

where $h(x) = \left[e^{-2x}/(1+e^{-2x})\right] + \frac{c}{\theta^2}x(1-e^{-2x})/(1+e^{-2x})$ and $Q = \frac{1}{2}P_{-\theta} + \frac{1}{2}P_\theta$.

For $x > 0$ the function h is convex and has a minimum in $b^*(c)$. Thus $R(T,\delta) \geq h(b^*(c))$. Let $T^* = \inf\{t > 0 \mid \theta|W(t)| \geq b^*(c)\}$ denote the stopping time, which stops in the minimum of h. Then (T^*, δ^*_T) minimizes the Bayes risk $(*)$.

The structure given in $(*)$ is also present when testing composite hypotheses, for certain change-point detection problems and for other testing problems with composite hypotheses. In the case of discrete observations one cannot stop in the minimum with probability 1 and one has to consider overshoot corrections.

3. The second example discusses a classical stopping problem: Let W denote Brownian motion with $M_0 = 1$. Then

$$E_{x_0}\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) = \max!$$

is to maximize over all stopping times. Let $H(x) = \int_0^\infty e^{ux-u^2/2}u^{2\beta-1}du$ with $\beta > 0$ and assume that there exists a unique point x^* with

$$\sup_{x\in\mathbb{R}} \frac{g(x)}{H(x)} = \frac{g(x^*)}{H(x^*)} = C^* \text{ and } 0 < C^* < \infty.$$

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Then $M_t = (t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right) / H(x_0)$ is a positive martingale with starting value 1 and further

$$E_{x_0}\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) = H(x_o) E_{x_0} \frac{g\left(\frac{X_T}{\sqrt{T+1}}\right)}{H\left(\frac{X_T}{\sqrt{T+1}}\right)} M_T \leq H_0(x_0) C^*.$$

Let $T^* = \inf\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\}$, then if $x_0 < x^*$ it holds $P_{x_0}(T^* < \infty) = 1$ and $EM_{T^*} = 1$. Thus T^* is optimal.

For the special case: $h(x) = x$, $x_0 = 0$, and $\beta = \frac{1}{2}$ one has

$$E(X_T/(T+1)) = \max!$$

Then x^* is the solution of $x = (1-x^2)\int_0^\infty e^{-ux-u^2/2} du$, a result once derived by L. Shepp.

In general, let X denote a diffusion process and consider the problem to solve

$$V(x) = \sup_\tau E_x e^{-r\tau} g(X_\tau),$$

where x denote the starting point of X.

We suggest to find a positive function h such that $M_t = e^{-rt} h(X_t)$ is a positive local martingale and $\sup_x \frac{g}{h}(x) = C^* < \infty$. Then

$$\begin{aligned} E_x e^{-r\tau} g(X_\tau) &= E_x\left(e^{-r\tau} h(X_\tau) \frac{g(X_\tau)}{h(X_\tau)}\right) \\ &\leq C^* E_x e^{-r\tau} h(X_\tau) \\ &\leq C^* h(x). \end{aligned}$$

If we can find a stopping time τ^* with $\frac{g}{h}(X_{\tau^*}) = C^*$ and $E_x(e^{-r\tau^*} h(X_{\tau^*})) = h(x)$, then the inequalities become equalities and the optimal stopping problem has as solution $V(x) = C^* h(x)$.

We shall give several examples of this method and shall characterize with it the optimal stopping set $\{V = g\}$ in a more concrete way.

4. In the third part we consider stopping games, which can be interpreted as options in the sense of Kifer. We extend the approach described above to give sufficient conditions for Nash-equilibria of such games. This extension uses appropriate harmonic functions which are neither convex nor concave.

References

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