# Stochastic control and free boundary problem for sailboat trajectory optimization

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I will present a stochastic control problem motivated by sailing races. The goal is to minimize the travel time between two locations, by selecting the fastest route in face of randomly changing weather conditions, such as wind direction. When a sailboat is travelling upwind, the key is to decide when to tack, that is, to switch bearings in such a way that if the wind was coming from one side of the yacht before the tacking, then, after the tacking, it comes from the other side of the yacht. Since this maneuver slows down the yacht, it is natural to model this time lost by a *tacking penalty*, $c > 0$. This places the problem in the context of optimal stochastic control problems with switching costs.

Our model preserves some of the real-world features of wind variability while eliminating some of the geometric problems arising from the specifics of yacht motion. We assume that the *yacht's speed* $v$ is a constant function of the angle $\gamma$ between the yacht's bearing and the wind direction: $v(\gamma) = v\,\mathbf{1}_{\{|\gamma| \geq \frac{\pi}{4}\}}$, $\gamma \in [-\pi, \pi]$. This is a big simplification which still preserves the main features of the initial problem. Indeed, though the speed of the yacht certainly depends on $\gamma$, assuming that sailing settings are chosen so as to maximize the yacht's upwind velocity, the yacht sails mostly at a nearly constant speed. Furthermore, in order to simplify the problem, we consider that the wind speed is also constant and that only the *wind direction* $(W_t)_{t\geq 0}$ is random. We assume that $(W_t)$ is a two-state continuous-time Markov chain defined on a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in $\{\pm\alpha\}$ with $\alpha \in ]0, \frac{\pi}{4}[$. A *tacking strategy* is defined as a right continuous, piecewise constant and adapted process $(A_t)_{t\geq 0}$ taking values in $\{\pm 1\}$. If $A_t = 1$ (resp. $-1$), it means that at time $t$, the yacht is sailing on starboard (resp. port) tack. Namely, the wind enters the sails from its right (resp. left) side. The *number of tackings* of a strategy $A$ is given by the following process:

$$N_t(A) = \sharp\{s \in [0, t] : A_s \neq A_{s-}\}.$$

To ensure model consistency, a strategy is *admissible* only if it satisfies some extra conditions which will be given during the presentation. If one starts at a point $\vec{x}$ of the race area, on a tack $a \in \{\pm 1\}$ and under a wind $w \in \{\pm\alpha\}$, then the *payoff function* of a race driven by an admissible tacking strategy $A$ is given by

$$J(\vec{x}, a, w, A) = \mathbb{E}_{\vec{x},a,w}\left(\tau^A + cN_{\tau^A}(A)\right)$$

where $\tau^A$ is the hitting time of the target buoy. The *value function* of the problem is then given by

$$V(\vec{x}, a, w) = \inf_{A \text{ admiss.}} J(\vec{x}, a, w, A).$$

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First, we will discuss some properties of the solution and the concept of a *lifted tack*, specific to route optimization. This allows us to characterize the wind into two categories (stable or unstable) related to the mean time between changes in wind direction. Several asymptotic cases have been studied in [2]. Here, I would like to present a particular case where it is possible to find an explicit solution of the problem. We assume that the yacht starts close to the target buoy under a stable wind. In this case, we can show that the value function solves a system of first order partial differential equations with free boundaries that are easy to find. The system can be transformed into second order hyperbolic partial differential equations of Klein-Gordon type. We compute explicitly the solutions of these equations, which give formulas for the value function of the problem. A verification theorem establishes then the optimality of the solution. I will conclude by giving the general shape of the solution when we consider the problem in the entire state space and the procedure to compute the value function in that case.

This work has been done in collaboration with R. Dalang and was highly motivated by the work of F. Dumas in [1].

## References

[1] F. Dumas (2012). Stochastic Optimization of Sailing Trajectories in an America's Cup Race. *PhD thesis 4884, École Polytechnique Fédérale de Lausanne.*

[2] L. Vinckenbosch (2012). Stochastic Control and Free Boundary Problems for Sailboat Trajectory Optimization. *PhD thesis 5381, École Polytechnique Fédérale de Lausanne.*