A general Bayesian disorder problem for a Brownian motion on a finite interval

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1. Suppose we sequentially observe a process $X = (X_t)_{t\ge 0}$ satisfying the equation

$$dX_t = \mu \mathbf{I}(t \ge \theta)dt + dB_t, \qquad X_0 = 0,$$

where $B = (B_t)_{t\ge 0}$ is a standard Brownian motion, θ is a non-negative *unobservable* random variable taking values in an interval $[0, T]$ with a known distribution, independent of B, and $\mu > 0$ is a known number. The random variable θ can be interpreted as the moment of disorder – the moment when the drift of X changes. We consider the problem of detecting the disorder that consists in finding the stopping time τ^* of the filtration $(\mathcal{F}_t^X)_{t\ge 0}$ which is "as close as possible" to θ.

Let $H(t)$ be *a penalty function*, which decreases for $t < 0$, increases for $t > 0$ and $H(0) = 0$. Mathematically, we look for the stopping time τ^* such that

$$\mathsf{E}H(\tau^* - \theta) = \inf_{\tau} \mathsf{E}H(\tau - \theta), \tag{1}$$

where the infimum is taken over all stopping times τ of $(\mathcal{F}_t^X)_{t\ge 0}$.

Some particular cases of problem (1) have been considered in the literature mainly when θ is exponentially distributed and $H(t)$ is of a special form (see the review in [1]). In the present paper we provide a general solution to the problem when θ takes values in a finite interval $[0, T]$.

2. We consider the case when $H(t)$ is linear or exponential for $t \ge 0$, i. e.

$$H(t) = ct \ \text{ for } t \ge 0 \qquad \text{or} \qquad H(t) = \frac{c}{b}e^{bt} \ \text{ for } t \ge 0,$$

where $b, c > 0$ are known numbers. In the linear case, for convenience, we assume $b = 0$. Introduce *the generalized Shiryaev–Roberts statistic* $\psi = (\psi_t^{(b)})_{t\ge 0}$:

$$\psi_t^{(b)} = e^{\mu X_t - (\mu^2/2 - b)t} \int_0^t e^{-\mu X_s + (\mu^2/2 - b)s} dG(s).$$

Let E^∞ denote the mathematical expectation with respect to the measure, under which X is a Brownian motion, and define function $\tilde{H}(t) = \int_t^\infty H(t-s)dG(s)$.

We show that under mild smoothness conditions on G, the optimal stopping time τ^* in problem (1) can be found as the first hitting time of $\psi^{(b)}$ to a time-dependent level:

$$\tau^* = \inf\{t \ge 0 : \psi_t^{(b)} \ge a(t)\} \wedge T,$$

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where the function $a\colon [0,T] \to \mathbb{R}_+$ is the unique continuous solution of the equation

$$\int_t^T \mathsf{E}^\infty \big[(c\psi_s^{(b)} + \tilde{H}'(s))\mathbf{I}\{\psi_s^{(b)} < a(s)\} \,\big|\, \psi_t^{(b)} = a(t)\big] ds = 0, \quad t \in [0,T],$$

satisfying the conditions

$$a(t) \geq -\tilde{H}'(t)/c \text{ for } t < T, \qquad a(T) = -\tilde{H}'(T-)/c.$$

The average penalty $\mathcal{H} = \mathsf{E}H(\tau^* - \theta)$ can be found by the formula

$$\mathcal{H} = \tilde{H}(0) - \int_0^T \mathsf{E}^\infty \big[(c\psi_s^{(b)} + \tilde{H}'(s))\mathbf{I}\{\psi_s^{(b)} < a(s)\}\big] ds.$$

3. The main idea consists in reducing problem (1) to the following optimal stopping problem for the process $\psi^{(b)}$:

$$V = \inf_{\tau \leq T} \mathsf{E}^\infty \left[\tilde{H}(\tau) + c\int_0^\tau \psi_s^{(b)} ds\right].$$

The solution of this problem is found using standard methods.

References

[1] Shiryaev A. N. (2010). Quickest Detection Problems: Fifty Years Later. *Sequential Analysis* 29, 345–385.

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